Electricity and magnetism – From Gauss's law to the Maxwell-Equations

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Electric field, magnetic field, dielectric displacement, magnetic flux, Coulomb-force, principle of super-position, Biot and Savart's law, Ampères circuital law, current conservation (continuity equation), magnetic monopoles do not exist, Faraday's law of induction, Maxwell'sdisplcaement current and Maxwell's Equations, potentials, Lorentz force, boundary conditions of the electric displacement, electric field, magnetic induction and magnetic field, field of a dipole, force and torque on a dipole in a background field.

Historical evolution

Compass	presumably China 2500 B.C.
Loadstone attracts iron	600 B.C.
Magnetite can magnetize iron	Ancient Greece
Earth possesses a magnetic field	Gilbert (1540-1603)
Discovery of magnetic poles	Peregrines (1269)
Forces which act on poles	Coulomb (1736-1806)
Magnetism as moving electricity	Ampère (1822)
Field equations	Maxwell (1862)
Modern formulation of the field equations	Hertz (1890)
Special theory of relativity	Einstein (1905)

Coulomb's law and the electric field E

Law describing the forces between two electric charges (poles)

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12}$$

F can be seperated into a testing pole q_2 and a field, which causes the force acting on the testing pole

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r}_0$$

$$[E] = 1\,\mathrm{V/m}$$

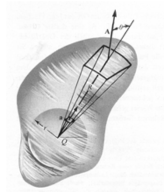
Gauss's law

The total charge in a volume is found by integration

$$Q = \int_V dQ = \int_V \rho\, dv$$

To an electric charge an electric flux of an absolute value of 1 C is dedicated:

$$\Psi = Q(C)$$



The electric displacement density is then given by:

$$d\Psi = \vec{D}\, d\vec{S}$$

Gauss's law states, that:

$$\oint_{\partial V} \vec{D}\, d\vec{S} = Q$$

Integrating Coulomb's law, we find that: $\vec{D} = \varepsilon_0 \vec{E}$

To find Gauss's law in ist differential form, we use the divergence theorem:

Accordingly to Gauss's law, it holds that

$$\frac{\oint \vec{D}\cdot d\vec{S}}{\Delta \mathrm{v}} = \frac{Q_{\mathrm{enclosed}}}{\Delta \mathrm{v}}$$

with Lim $\Delta \mathrm{v} \to 0$ the left side of the equation converges to div **D**, whereas the right side converges to ρ .

We therefore get one of the Maxwell equations for stationary electric fields, namely

$$\mathrm{div}\vec{D} = \rho \quad \text{or} \quad \mathrm{div}\vec{E} = \rho/\varepsilon$$

From Gauss's law, it follows the divergence theorem, namely

$$\oint \vec{D}\cdot d\vec{S} = \oint \mathrm{div}\vec{D}\, dV$$

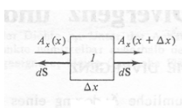
On the other hand, starting from the Maxwell equation $\mathrm{div}\vec{D} = \rho$ and using the divergence theorem, one obtains Gauss's law $\oint_{\partial V} \vec{D}\, d\vec{S} = \int \rho\, dV$.

We compute the divergence in Cartesian coordinates. Let **A** be a vectorfield:

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$

We compute the surface Integral $\oint \vec{A}\cdot d\vec{S}$ by summing up over a cube where at the left side we have

$$\int_{\text{left side}} \vec{A}\cdot d\vec{S} \approx -A_x(x)\Delta y \Delta z$$



And on the right side, we have

$$\int_{\text{right side}} \vec{A}\cdot d\vec{S} \approx A_x(x+\Delta x)\Delta y \Delta z$$

$$\approx \left[A_x(x) + \frac{\partial A_x}{\partial x}\Delta x\right]\Delta y \Delta z$$

So that from these two sides we have $\frac{\partial A_x}{\partial x}\Delta x \Delta y \Delta z$

Adding up the contributions from all sides, we obtain

$$\oint \vec{A}\cdot d\vec{S} \approx \left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

so that by the definition of the divergence

$$\operatorname{div} \vec{A} = \lim_{\Delta v \to 0} \frac{\oint \vec{A}\cdot d\vec{S}}{\Delta v}$$

we get

$$\operatorname{div}\vec{A} = \left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)$$

Magnetic monopoles do not exist

Isolated magnetic monopoles never have been observed in nature. Magnetic poles always appear as pairs. A pair of poles forms a dipole. The magnetic moment of the dipole is given by

$$\vec{\mu} = m\,\vec{d}$$

d shows from the negative to the positive pole.

Stated mathematically: The magnetic field has no sources. This is also expressed by the fact, that the magnetic field lines are always closed loops. Therefore, there's no Gauss law for the magnetic induction. Instead it holds that:

$$\oint_{\partial V} \vec{B}\, d\vec{S} = 0$$

Or, otherwise stated while applying the divergence theorem for arbitrary volumes:

$$\operatorname{div} \vec{B} = 0$$

that means that for infinitesimal volumes there is no net flux coming out of such a volume, i.e. there are no magnetic charges.

Ampère's law and Biot and Savart's law

In case of time-independent (stationary) currents, Ampère's law holds

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 \int \vec{J}\cdot d\vec{A}$$

i.e. we assign to a stationnary current density **J** – analogous to Gauss's law – a magnetic flux **B**.

From Ampère's law and the absence of sources of the magnetic field it follows (e.g. using potential theory) the Biot and Savart's law:

$$d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{d\vec{\ell}\times\hat{r}}{r^2}$$

It follows – analogous to Ampère's law – from the observation of forces, which act on currents.

The limit of the relationship of circulation to loop area can be written for loops being orentatated according to the right hand rule as

$$\lim_{A_i\to 0}\frac{\oint_{C_i}\vec{F}\cdot d\vec{S}}{A_i}$$

which gives a scalar for each direction **x**, **y**, **z**, which are components of a vector called curl

$$(\operatorname{curl} F)\cdot\hat{n} = \lim_{A_i\to 0}\frac{\oint_{C_i}\vec{F}\cdot d\vec{S}}{A_i}$$

We return to the original loop C, where we get the circulation

$$\Gamma = \int_C \vec{F}\cdot d\vec{S} = \sum_{i=1}^{n}\Gamma_i = \sum_{i=1}^{n} A_i\left[\frac{\Gamma_i}{A_i}\right] = \sum_{i=1}^{n} A_i(\operatorname{curl}\vec{F})\cdot\hat{n}_i \to \int_A d\vec{A}\cdot\operatorname{curl}\vec{F}$$

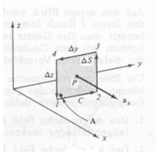
This is the theorem of Stokes

$$\int_C \vec{F}\cdot d\vec{S} = \int_A \operatorname{curl}\vec{F}\cdot d\vec{A}$$

From which we get the Ampere's law in ist differential form

$$\operatorname{curl}\vec{B} = \mu_0\vec{J}$$

We now compute the curl of **F** in Cartesian coordinates



$$(\text{curl } F)\cdot a_x = \lim_{\Delta y \Delta z \to 0} \frac{\oint \vec{F}\cdot d\vec{l}}{\Delta y \Delta z}$$

We consider an infinitesimal area oriented in direction $\mathbf{a}_x$

$$\oint = \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1$$

$$= F_y \Delta y + \left(F_z + \frac{\partial F_z}{\partial y}\Delta y\right)\Delta z + \left(F_y + \frac{\partial F_y}{\partial z}\Delta z\right)(-\Delta y) + F_z(-\Delta z)$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\Delta y \Delta z$$

The y and z components are computed correspondingly. Together we obtain

$$\text{curl } \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{a}_z$$

Electromagnetic potentials

The electric field in the absence of charges is conservative, so that **E** can be derived from a scalar potential

$$\vec{E} = -\text{grad}\,\Phi$$

For a single charge it holds that

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

This is a solution of the Poisson equation

$$\nabla\cdot\nabla\Phi = -\rho/\varepsilon$$

A general solution would be

$$\Phi(r_1) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r_2)}{r_{12}} dV_2$$

The divergence of the magnetic field vanishes, so that **B** can be written as curl of a voctor potential

$$\vec{B} = \text{curl}\,\vec{A}$$

Ampère's law

$$\text{curl}(\,\text{curl}\,\vec{A}) = \mu_0 \vec{J}$$

Can be rewritten as

$$\text{grad}(\,\text{div}\,\vec{A}) - \text{div}(\text{grad}\,\vec{A}) = \mu_0 \vec{J}$$

Using the Coulomb gauge

$$\text{div}\,\vec{A} = 0 \qquad \vec{A} \to \vec{A} + \nabla\Psi$$

this results in the Poisson equation for **A**

$$\nabla\cdot\nabla\vec{A} = -\mu_0 \vec{J}$$

The solution we do already know from the electric case

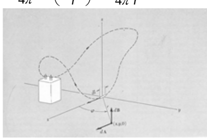
$$\vec{A}(r_1) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r_2)}{r_{12}} dV_2$$

with $d\vec{B} = \text{curl}\; d\vec{A}$ we obtain

$$d\vec{B} = \nabla\times\frac{\mu_0 I}{4\pi}\frac{d\vec{l}}{r} = -\frac{\mu_0 I d\vec{l}}{4\pi}\times\nabla\frac{1}{r}$$

$$= -\frac{\mu_0 I d\vec{l}}{4\pi}\times\left(-\frac{\hat{r}}{r^2}\right) = \frac{\mu_0 I d\vec{l}\times\hat{r}}{4\pi\, r^2}$$

this is Biot and Savart's law.



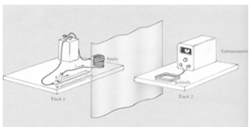
Faraday's law of induction

We have seen: Flowing electricity causes a magnetic field. On the other hand do time-dependent magnetic fields (a change in the magnetic flux through a conducting loop) lead to an induced Voltage and thereby cause an electric field. The penomenon of induction observed by Faraday can be stated as follows:

$$\oint_C \vec{E}\, d\vec{s} = -\frac{d\Phi}{dt} = -\int \frac{d\vec{B}}{dt} d\vec{A}$$

Or, using the theorem of Stokes,

$$\text{curl}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$



Magnetic fields in matter: Magnetisation M und polarisation P

A material in a magnetic fieldexperiences a magnetisation **M**, defined as Dipole moment per unit volume.

It holds Ampère's law (differential form)

$$\text{curl}\,\vec{B} = \mu_0 \vec{j}$$

where **j** is a surface current density. The magnetisation **M** thus possesses an equivalent current density

$$\vec{j}_M = \text{curl}\,\vec{M}$$

To fix **B** we must evaluate this equivalent current density

$$\text{curl}\,\vec{B} = \mu_0(j + j_M) \text{ or } \text{curl}(\vec{B} - \mu_0 M) = \mu_0 j$$

Thus write $\mu_0\vec{H} = \vec{B} - \mu_0\vec{M}$

and it follows $\vec{B} = \mu_0(\vec{H} + \vec{M})$

In the same way a material in an electric field experiences a polarisation **P**:

A material in an electric fieldexperiences a polarisation **P**, defined as Dipole moment per unit volume.

It holds Gauss's law (differential form)

$$\mathrm{div}\,\vec{D} = \rho$$

where ρ is a volume charge density. The polarisation **P** thus possesses an equivalent charge density

$$\rho_p = -\,\mathrm{div}\,\vec{P}$$

To fix **E** we must evaluate this equivalent charge density

$$\mathrm{div}\,\vec{E} = \rho + \rho_p \quad \text{or} \quad \mathrm{div}\left(\vec{E} + \vec{P}\right) = \rho$$

Thus write $\varepsilon_0 \vec{E} = \vec{D} - \vec{P}$

and it follows $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

Boundary conditions

The boundary conditions for the surface between two dielectrica say that

- the tangential component of **E** is continuous across the boundary

$$E_{t_1} - E_{t_2} = 0$$

- the normal component of **D** makes a step of the amount of the absolute value of the surface current density

$$\left(\vec{D}_2 - \vec{D}_1\right) \cdot \hat{n} = \rho$$

Boundary conditions

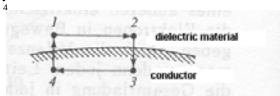
the electric field is conservative

$$\oint \vec{E}\, d\vec{l} = \int_1^2 \vec{E}\, d\vec{l} + \int_2^3 \vec{E}\, d\vec{l} + \int_3^4 \vec{E}\, d\vec{l} + \int_4^1 \vec{E}\, d\vec{l} = 0$$

$$\left.\begin{matrix} 2 \to 3 \\ 4 \to 1 \end{matrix}\right\} \Rightarrow 0$$

$3 \to 4$ inside the conductor $\to 0$

$$\int_1^2 \vec{E}\, d\vec{l} = \int_1^2 E_t dl = 0$$



It follows

$E_t = D_t = 0$ for a conductor;

$E_{t_1} = E_{t_2}$ for a dielectric material

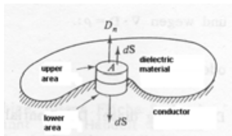
Boundary conditions

Starting from Gauss's law

$$\oiint \vec{D}\, d\vec{S} = \int_{\text{top area}} \vec{D}\, d\vec{S} + \int_{\text{bottom area}} \vec{D}\, d\vec{S} + \int_{\text{lateral area}} \vec{D}\, d\vec{S} = \int_A \rho_S\, d\vec{S}$$

The second integral disppears because of being inside the guide. The third integral disappears because of $D_t = 0$

$$\int_{\text{upper area}} \vec{D}\, d\vec{S} = \int_{\text{upper area}} D_n\, dS = \int \rho_S\, dS$$



It follows

$D_n = \rho_S$ for a conductor;

$D_{n_1} - D_{n_2} = \rho_S$ for a dielectric material

Boundary conditions

The boundary conditions for the magnetic fields at a boundary say that

- the tangential component of **H** makes a step at the boundary of the amount of the current density

$$\left(\vec{H}_1 - \vec{H}_2\right) \times \hat{n} = \vec{K}$$

- the normal component of **B** is continuous across the boundary

$$\left(\vec{B}_2 - \vec{B}_1\right) \cdot \hat{n} = 0$$

Boundary conditions

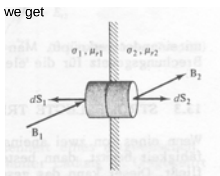
Starting from the non-existence of magnetic monopoles

$$\oiint \vec{B}\, d\vec{S} = \int_{\text{top area}} \vec{B}_1\, d\vec{S}_1 + \int_{\text{bottom area}} \vec{B}_2\, d\vec{S}_2 + \int_{\text{lateral area}} \vec{B}\, d\vec{S} = 0$$

Letting the high of the cylinder approaching zero we get

$$\oiint \vec{B}\, d\vec{S} = -\vec{B}_{n_1} \int_{\text{top area}} d\vec{S}_1 + \vec{B}_{n_2} \int_{\text{bottom area}} d\vec{S}_2 = 0$$

or $B_{n_1} = B_{n_2}$



Boundary conditions

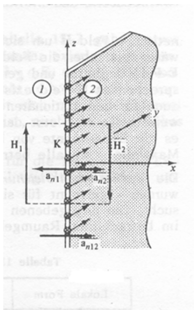
From Ampère's law we get (left for an exercise)

$$\vec{H}_1 = \frac{1}{2}\vec{K} \times \hat{a}_{n_1}$$

$$\vec{H}_2 = \frac{1}{2}\vec{K} \times \hat{a}_{n_2}$$

So that

$$\left(\vec{H}_1 - \vec{H}_2\right) \times \hat{a}_{n_{12}} = \vec{K}$$



Field of a dipole

The potential of a charge distribution

$$U_A = k\int \frac{\rho(r')dV'}{R} \qquad k = (4\pi\varepsilon_0)^{-1}$$

$$R = \left[r^2 + r'^2 - 2rr'\cos\theta\right]^{-1/2} = \frac{1}{r}\left[1 + \left(\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta\right)\right]^{-1/2}$$

$$\cong \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta + O\left(\frac{1}{r^3}\right)\right]$$

The potential in A is then given by

$$U_A = \frac{k}{r}\int \rho\, dV' + \frac{k}{r^2}\int r'\cos\theta\, \rho\, dV' + O\left(\frac{1}{r^3}\right)$$

Or, generally with $r'\cos\theta = \hat{r}\,\vec{r}'$ and with dipole moment defined as $\vec{p} = \int \vec{r}'\rho\, dV'$

$$U_A = \frac{k}{r^2}\int \hat{r}\,\vec{r}'\rho\, dV' = k\frac{\hat{r}\,\vec{p}}{r^2}$$

e.g. for a dipole in the origin, oriented in z-direction, we have a potential of

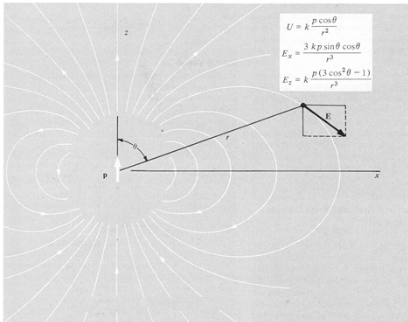
$$U = k\frac{p\cos\theta}{r^2} = k\frac{p\,z}{\left(x^2+z^2\right)^{3/2}}$$

so that the components of the elecric field are given by

$$E_x = -\frac{\partial U}{\partial x} = \frac{3k\,p\,x\,z}{\left(x^2+z^2\right)^{5/2}} = \frac{3k\,p\sin\theta\cos\theta}{r^3}$$

$$E_z = -\frac{\partial U}{\partial z} = kp\left[\frac{3z^2}{\left(x^2+z^2\right)^{5/2}} - \frac{1}{\left(x^2+z^2\right)^{3/2}}\right]$$

$$= k\frac{p\left(3\cos^2\theta - 1\right)}{r^3}$$



Force and torque on a dipole

The torque of an electric dipole in an electric field is given by

$$\vec{T} = \vec{p} \times \vec{E}$$

The force on an electric dipole in an inhomogeneous field is given by

$$\vec{F} = (\vec{p}\cdot\nabla)\ \vec{E}$$

The far-fields of electric and magnetic dipoles are identical.

Lorentz force an a moving charge

The relation between fields, electric charges and currents in the general case (time-dependent case) is described by the general expression for the Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

There it appears the electric field **E**, as well as the magnetic flux **B**.

Conservation of charge

We design by **j** the current density of the moving charge distribution

$$\vec{j} = \rho \vec{v}$$

For reasons of conservation of charge and from the Gauss law applied on a infinitesimal volume it follows the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$$

Since charges can be neither created nor annihilated, the continuity equation for charge density ρ and current density **j** holds: $\operatorname{div} \vec{j} = -\frac{\partial \rho}{\partial t}$

For a stationary current density **j** the magnetic field obeys Ampère's law: $\operatorname{curl} \vec{B} = \mu_0 \vec{j}$

Looking at time-dependent charge densities and fields $\rho(\vec{x}, t)$ with $\partial\rho/\partial t \neq 0$

e.g. a condensator decharging via a resistor, the continuity equation tells us that

$$\operatorname{div} \vec{j} \neq 0$$

From Ampère's law, however, it always is true that

$$\operatorname{div} \vec{j} = \frac{1}{\mu_0} \operatorname{div} (\operatorname{curl} \vec{B}) = 0$$

Ampère's law thus is not true for the case of a time-varying charge density.

In the general case of time-varying charge densites we write

$$\operatorname{curl} \vec{B} = \mu_0 \vec{j} + (?)$$

We recall the differential formulation of Faraday's induction law

$$\operatorname{curl} \vec{E} = -\frac{1}{c} \frac{\partial c\vec{B}}{\partial t}$$

This is a local relation between the electric and the magnetic field if free-space charges are not concerned.

For symmetry reasons between **E** and c**B**, we use a kind of induction phenomenon

$$\operatorname{curl} c\vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \Leftrightarrow \operatorname{curl} \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

We write

$$\operatorname{curl} \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Which is another Maxwell Equation.

Taking the divergence on both sides leads to

$$\operatorname{div} (\operatorname{curl} \vec{B}) = \operatorname{div}(\mu_0 \vec{j}) + \frac{1}{c^2} \frac{\partial}{\partial t} \operatorname{div} \vec{E}$$

And, because $\operatorname{div} \operatorname{curl} = 0$,

$$-\mu_0 \operatorname{div}(\vec{j}) = \frac{1}{c^2} \frac{\partial}{\partial t} \operatorname{div} \vec{E} = \mu_0 \frac{\partial \rho}{\partial t}$$

or

$$\operatorname{div}(\vec{j}) + \frac{\partial \rho}{\partial t} = 0$$

which again is the continuity equation.

The term

$$\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Is called displacement current and is needed for the continuity equation to be fulfilled.

The Maxwell equations

From the gauss law, the absence of magnetic monopoles, Faraday's law of induction and Ampère's law follows by taking into account the continuity equation the Maxwell-equations (in its differential form)

$$\operatorname{div} E = \rho/\varepsilon_0 \qquad \operatorname{curl} \vec{E} = -d\vec{B}/dt$$

$$\operatorname{div} B = 0 \qquad \operatorname{curl} \vec{B} = \mu_0 (\vec{j} + \varepsilon_0 \, d\vec{E}/dt)$$

Maxwell introduced the displacement current (dielektric displacement), to obtain consistency.

From the Maxwell-equations (ME) it follows the existence of electromagnetic waves as well as the special theory of relativity (STR).

Recommended reading:

E.M. Purcell Electricity and Magnetism: Berkeley Physics Course, Vol. 2

J.D. Jackson Classical Electrodynamics 2nd. Ed.